

AN EXTREMAL PROBLEM ON CROSSING VECTORS

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ABSTRACT. For positive integers w and k , two vectors A and B from \mathbb{Z}^w are called k -crossing if there are two coordinates i and j such that $A[i] - B[i] \geq k$ and $B[j] - A[j] \geq k$. What is the maximum size of a family of pairwise 1-crossing and pairwise non- k -crossing vectors in \mathbb{Z}^w ? We state a conjecture that the answer is k^{w-1} . We prove the conjecture for $w \leq 3$ and provide weaker upper bounds for $w \geq 4$. Also, for all k and w , we construct several quite different examples of families of desired size k^{w-1} . This research is motivated by a natural question concerning the width of the family of maximum antichains of a partially ordered set.

1. INTRODUCTION

Let \mathbb{Z} denote the set of integers. For a positive integer w , let \mathbb{Z}^w denote the cartesian product of w copies of \mathbb{Z} . Elements of \mathbb{Z}^w are called *vectors*. The i th coordinate of a vector $A \in \mathbb{Z}^w$ is denoted by $A[i]$, for $1 \leq i \leq w$. The product ordering on \mathbb{Z}^w is defined by setting $A \leq B$ for $A, B \in \mathbb{Z}^w$ if $A[i] \leq B[i]$ for every coordinate i .

When $k \geq 1$, we say that vectors A and B from \mathbb{Z}^w are k -crossing if there are coordinates i and j for which $A[i] - B[i] \geq k$ and $B[j] - A[j] \geq k$. Thus \mathcal{A} is an antichain in \mathbb{Z}^w if and only if any two distinct vectors from \mathcal{A} are 1-crossing.

For positive integers k and w , let $f(k, w)$ denote the maximum size of a subset of \mathbb{Z}^w with any two vectors being 1-crossing but not k -crossing. In other words, $f(k, w)$ is the maximum size of an antichain in \mathbb{Z}^w with no two k -crossing vectors.

Determining the value of $f(k, w)$ is the main focus of this paper. We find the exact value for $w \leq 3$ and provide upper and lower bounds for $w \geq 4$. Still we are unable to resolve the following striking simple conjecture.

Conjecture 1 (Felsner, Krawczyk, Micek 2010). *For all $k, w \geq 1$ we have*

$$f(k, w) = k^{w-1}.$$

Theorem 2. *For all $k, w \geq 1$ we have*

$$k^{w-1} \leq f(k, w) \leq k^w - (k-1)^w.$$

Theorem 3. *For $1 \leq w \leq 3$ and $k \geq 1$ we have*

$$f(k, w) = k^{w-1}.$$

The remainder of this paper is organized as follows. We start, in the next section, by a brief discussion of problems in partially ordered sets that initiated this line of research. Sections 3 and 4 are devoted to the proofs of Theorems 2 and 3, respectively. In Section 5 we propose another conjecture, which is at first glance more general but in fact equivalent to Conjecture 1. Concluding in Section 6, we provide examples of families witnessing $f(k, w) \geq k^{w-1}$ with a discussion why the

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full resolution of the conjecture seems to be difficult. We also present a proof of the conjecture for families of vectors with a single coordinate differentiating all vectors in the family and another argument for ranked families of vectors, that is, families in which the coordinates of every vector sum up to the same value.

2. BACKGROUND MOTIVATION

Let $\mathcal{M}(P)$ denote the family of all maximum antichains (that is, antichains of maximum size) in a finite poset P . The family $\mathcal{M}(P)$ is partially ordered by setting $A \leq B$ when for every $a \in A$ there is $b \in B$ with $a \leq b$ in P . The family $\mathcal{M}(P)$ equipped with this partial order forms a lattice. In the following, we are concerned about the order structure of $\mathcal{M}(P)$, in particular its width.

For a positive integer k let $\mathbf{k} + \mathbf{k}$ denote the $2k$ -element poset consisting of two disjoint k -element chains with no comparabilities between the points in distinct chains. Let $\mathcal{P}(k)$ denote the class of posets containing no subposet isomorphic to $\mathbf{k} + \mathbf{k}$. The posets in $\mathcal{P}(1)$ are just the chains, while $\mathcal{P}(2)$ is exactly the class of interval orders (see p. 86 of [5]). For positive integers k and w let $\mathcal{P}(k, w)$ denote the subclass of $\mathcal{P}(k)$ consisting of posets of width at most w .

Recently, several results in combinatorics of posets showed that problems that are difficult or even impossible to deal with for all posets of bounded width become much easier when only posets from $\mathcal{P}(k, w)$ are considered. This includes the on-line chain partitioning problem [1, 2, 4] and the on-line dimension problem [3].

It turns out that the width of $\mathcal{M}(P)$ may be bounded if and only if the width of P is bounded and the size of $\mathbf{k} + \mathbf{k}$ type structure in P is bounded as well. Namely, every upper bound on $f(k, w)$ yields the same upper bound on the width of $\mathcal{M}(P)$ for $P \in \mathcal{P}(k + 1, w)$, for any $k, w \geq 1$. On the other hand, the width of $\mathcal{M}(\mathbf{k} + \mathbf{k})$ is k , and the width of $\mathcal{M}(\underbrace{\mathbf{2} + \dots + \mathbf{2}}_w)$ is $\binom{w}{\lfloor w/2 \rfloor}$.

Conjecture 4 (Felsner, Krawczyk, Micek 2010). *Let k and w be positive integers with $k \geq 2$. The maximum width of $\mathcal{M}(P)$ for a poset $P \in \mathcal{P}(k, w)$ is $(k - 1)^{w-1}$.*

This conjecture was made prior to the formulation of Conjecture 1. It is easy to see that the positive resolution of Conjecture 1 will imply Conjecture 4. In particular, it follows from Theorem 3 that Conjecture 4 is true for $w \leq 3$. In the special case $k = 2$, the class $\mathcal{P}(2, w)$ is the class of interval orders of width w , and the conjecture states the well-known fact that maximum antichains in an interval order form a chain. Moreover, we are able to prove that Conjecture 4 for $k = 3$ and Conjecture 1 for $k = 2$ are equivalent.

3. GENERAL BOUNDS

The purpose of this section is to give the proof of Theorem 2, namely, that for all $k, w \geq 1$ we have

$$k^{w-1} \leq f(k, w) \leq k^w - (k - 1)^w.$$

Note that $f(k, 1) = k^0 = 1$ for every $k \geq 1$, as all antichains in \mathbb{Z}^1 are of size 1. Also $f(1, w) = 1^{w-1} = 1$ for every $w \geq 1$, as in this case we require every pair of distinct vectors to be simultaneously 1-crossing and not 1-crossing.

For the lower bound, observe that the following family is an antichain in \mathbb{Z}^w , contains no two k -crossing vectors, and has size k^{w-1} :

$$\{A \in \mathbb{Z}^w : 0 \leq A[i] \leq k - 1 \text{ for } 1 \leq i \leq w - 1, \text{ and } A[1] + \dots + A[w] = 0\}.$$

For the upper bound, we start by an easy argument that yields the bound of k^w . Let \mathcal{A} be an antichain in \mathbb{Z}^w containing no two k -crossing vectors. For each vector $A \in \mathcal{A}$, let $\sigma(A)$ be the vector from $\{0, \dots, k - 1\}^w$ such that $A[i] \equiv \sigma(A)[i]$

(mod k) for $1 \leq i \leq w$. If $\sigma(A) = \sigma(B)$ for distinct vectors $A, B \in \mathcal{A}$ and i, j are coordinates such that $A[i] > B[i]$ and $B[j] > A[j]$, then these two coordinates witness that A and B are k -crossing. It follows that σ is an injection. Since the size of the range of σ is k^w , we have $|\mathcal{A}| \leq k^w$.

We obtain better upper bounds using the following recursive formula.

Claim 5. *For $w \geq 2$ and $k \geq 1$ we have*

$$f(k, w) \leq k^{w-1} + (k-1)f(k, w-1).$$

Proof. Consider an antichain \mathcal{A} in \mathbb{Z}^w with no two k -crossing vectors. Consider the mapping $\phi : \mathcal{A} \rightarrow \mathbb{Z}^{w-1}$ truncating the last coordinate of every vector in \mathcal{A} , that is, such that $\phi(A)[i] = A[i]$ for $A \in \mathcal{A}$ and $1 \leq i \leq w-1$. Since \mathcal{A} is an antichain, ϕ must be injective. Let $\mathcal{B} = \{\phi(A) : A \in \mathcal{A}\}$. For each residue class $(r_1, \dots, r_{w-1}) \in \{0, \dots, k-1\}^{w-1}$, the set of vectors $\phi(A)$ with $A \in \mathcal{A}$ and $\phi(A)[i] \equiv r_i \pmod{k}$ forms a chain, as otherwise some two of these vectors would be k -crossing. Let \mathcal{M} denote the set of maximal elements of nonempty chains over all residue classes. Clearly, $|\mathcal{M}| \leq k^{w-1}$.

Now consider the poset formed by the vectors in $\mathcal{B} - \mathcal{M}$. We claim that the height of this poset is at most $k-1$. Once this fact has been established, it follows that $\mathcal{B} - \mathcal{M}$ can be partitioned into $k-1$ antichains. Each of these antichains has size at most $f(k, w-1)$ and the desired inequality follows: $|\mathcal{A}| \leq |\mathcal{M}| + (k-1)f(k, w-1)$.

So suppose to the contrary that $\mathcal{B} - \mathcal{M}$ has a chain $C_1 < \dots < C_k$ of k vectors. It follows that

$$\begin{aligned} C_1[i] &\leq \dots \leq C_k[i] \quad \text{for } 1 \leq i \leq w-1, \\ C_1[w] &> \dots > C_k[w]. \end{aligned}$$

Choose a vector $M \in \mathcal{M}$ so that $M[i] \equiv C_k[i] \pmod{k}$ for $1 \leq i \leq w-1$. Then $M[i] \geq C_k[i]$ for $1 \leq i \leq w-1$, and there is i_0 with $1 \leq i_0 \leq w-1$ for which $M[i_0] \geq k + C_k[i_0]$. It follows that

- (i) $M[i_0] - C_1[i_0] \geq M[i_0] - C_k[i_0] \geq k$;
- (ii) $M[w] < C_k[w]$ and therefore $M[w] - C_1[w] \geq k$.

This means that C_1 and M are k -crossing. The contradiction completes the proof. \square

From Claim 5 and the fact that $f(k, 1) = 1$ it follows that $f(k, w) \leq k^w - (k-1)^w$. This bound is better than both k^w and wk^{w-1} . By applying the recursive formula and Theorem 3 we get

$$f(k, w) \leq k^w - k^2(k-1)^{w-2}.$$

In a similar way as Claim 5 we can also prove that

$$f(k, w) \leq k^{w-v}f(k, v) + k^v f(k, w-v), \quad \text{for } 1 \leq v \leq w.$$

In view of the equality $f(k, 3) = k^2$, the latter gives an upper bound

$$f(k, w) \leq \frac{wk^{w-1}}{3}, \quad \text{for } w \geq 3.$$

4. THE CASE $w \leq 3$

In this section we prove Theorem 3, namely, that for $1 \leq w \leq 3$ and $k \geq 1$ we have

$$f(k, w) = k^{w-1}.$$

As explained at the beginning of the previous section, the equality holds for $w = 1$ or $k = 1$. Therefore, for the rest of this section, we assume that $2 \leq w \leq 3$ and $k \geq 2$. We only need to show that $f(k, w) \leq k^{w-1}$ as the converse inequality

is proved as part of Theorem 2. We start by the following easy proposition, stated for emphasis.

Proposition 6. *Let $w \geq 2$ and let \mathcal{A} be an antichain in \mathbb{Z}^w . If $S \subseteq \{1, \dots, w\}$, $|S| = w - 2$, and $A[i] = B[i]$ for every $A, B \in \mathcal{A}$ and every $i \in S$, then the two remaining coordinates $j, j' \in [w] - S$ determine two linear orders on \mathcal{A} , one dual to the other. That is, if we set $n = |\mathcal{A}|$, then there is a labelling A_1, \dots, A_n of the vectors in \mathcal{A} such that*

$$A_1[j] < \dots < A_n[j] \quad \text{and} \quad A_1[j'] > \dots > A_n[j'].$$

In particular, A_1 and A_n are $(n - 1)$ -crossing.

It follows immediately from Proposition 6 that $f(k, 2) \leq k$. Thus for the remainder of the argument we fix $w = 3$ and show that $f(k, 3) \leq k^2$ for $k \geq 2$.

We say that an antichain \mathcal{A} in \mathbb{Z}^w is *compressed* on the i th coordinate when $A[i] \geq 0$ for all $A \in \mathcal{A}$ and the quantity $\sum_{A \in \mathcal{A}} A[i]$ is minimized over all antichains of the same size. Let \mathcal{A} be an antichain in \mathbb{Z}^3 compressed on the third coordinate. It follows that $Q_3 = \{A[3] : A \in \mathcal{A}\}$ is an interval of non-negative integers starting from 0. By Proposition 6, the subfamily of \mathcal{A} consisting of all vectors A with $A[3] = s$ has size at most k for any $s \geq 0$. We conclude that $|\mathcal{A}| \leq k^2$ if $|Q_3| \leq k$. Thus for the remainder of the argument we assume $|Q_3| > k$.

Now we use coordinate 3 to define a directed graph D whose vertices are the vectors in \mathcal{A} . The edges in D are of two types: *short* and *long*.

- (i) D has a short edge from A to B when $A[3] - B[3] = 1$ and $A[i] \leq B[i]$ for $i \in \{1, 2\}$.
- (ii) D has a long edge from A to B when $B[3] - A[3] = k - 1$ and there is a coordinate $i \in \{1, 2\}$ for which $A[i] - B[i] \geq k$.

Claim 7. *For every $A \in \mathcal{A}$ there is a path (A_0, \dots, A_p) in D with $A_0 = A$ and $A_p[3] = 0$.*

Proof. The statement is trivial for $A \in \mathcal{A}$ with $A[3] = 0$. Suppose the conclusion of the claim is false for some vector $A \in \mathcal{A}$ with $A[3] > 0$. Let \mathcal{B} denote the subfamily of \mathcal{A} consisting of A and the vectors B in \mathcal{A} for which there is a directed path from A to B in D . Decrease coordinate 3 of each vector in \mathcal{B} by 1. The resulting family has the same size as \mathcal{A} , is an antichain, contains no two k -crossing vectors, and uses only non-negative coordinates. Moreover, the quantity q_3 has been decreased. This contradicts the choice of \mathcal{A} and completes the proof of the claim. \square

Claim 8. *For every $A \in \mathcal{A}$ with $A[3] \geq k$ there is a path (U_0, \dots, U_k) in D such that $U_0[3] = A[3]$ and (U_m, U_{m+1}) is a short edge in D for $0 \leq m \leq k - 1$.*

Proof. Fix $A \in \mathcal{A}$ with $A[3] \geq k$. For each $X \in \mathcal{A}$ with $X[3] = A[3]$, consider the length of the shortest path $X = (U_0, \dots, U_p)$ in D from X to a vertex U_p with $U_p[3] = 0$. Of all such X and U_p , take those for which the length p of the path is minimized. We show that the first $k + 1$ vectors on the chosen path satisfy the requirements of the claim. Suppose to the contrary that there is m with $0 \leq m \leq k - 1$ for which the edge (U_m, U_{m+1}) is long. Then $U_{m+1}[3] \geq A[3]$ and it follows that there is an integer n with $m + 1 \leq n < p$ for which $U_n[3] = A[3]$. This contradicts the choice of X and completes the proof of the claim. \square

In view of Claim 8, it is natural to refer to a path $\mathcal{P} = (U_0, \dots, U_p)$ in D as a *short path* when all edges on \mathcal{P} are short. Also, we say that the short edge (U, V) from D is *expanded in coordinate i* when $V[i] > U[i]$. Clearly, if (U, V) is a short edge in D , then it is expanded in one or both of coordinates 1 and 2 (as U and V are 1-crossing).

Let $\mathcal{A}_s = \{A \in \mathcal{A} : A[3] \equiv s \pmod{k}\}$ for $0 \leq s \leq k-1$. To complete the proof, we show that $|\mathcal{A}_s| \leq k$ for $0 \leq s \leq k-1$. Thus for the remainder of the argument we fix an integer s with $0 \leq s \leq k-1$. Let r be the largest integer for which there is a vector $A \in \mathcal{A}$ with $A[3] = s + (r-1)k$, and let $\mathcal{A}_s = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_r$ be the natural partition of \mathcal{A}_s such that $A[3] = s + (j-1)k$ for each $A \in \mathcal{B}_j$. We may assume that $r \geq 2$, as otherwise the conclusion that $|\mathcal{A}_s| \leq k$ follows from Proposition 6.

For $1 \leq j \leq r$, we refer to \mathcal{B}_j as *level j of \mathcal{A}_s* . Also, for $1 \leq j \leq r-1$, we apply Claim 8 and choose a short path \mathcal{P}_j of $k+1$ vectors starting at a vector $X_{j+1} \in \mathcal{B}_{j+1}$ and ending at a vector $Y_j \in \mathcal{B}_j$.

Claim 9. *For $2 \leq j \leq r-1$ we have $X_j \neq Y_j$.*

Proof. Suppose to the contrary that for some j with $2 \leq j \leq r-1$ we have $X_j = Y_j$. The ending point of the short path \mathcal{P}_j is the same as the starting point of the short path \mathcal{P}_{j-1} . It follows that the union of these two paths is a short path of $2k+1$ vectors starting at the vector X_{j+1} and ending at the vector Y_{j-1} . Denote the vectors on this path as $\mathcal{P} = (U_0, \dots, U_{2k})$, with $U_0 = X_{j+1}$ and $U_{2k} = Y_{j-1}$. We have

$$U_0[i] \leq \dots \leq U_{2k}[i] \quad \text{for } i \in \{1, 2\}.$$

Furthermore, for $0 \leq m \leq 2k-1$, the short edge (U_m, U_{m+1}) is expanded in some coordinate $i \in \{1, 2\}$. Since there are $2k$ short edges on \mathcal{P} , at least k of them are expanded in some coordinate $i \in \{1, 2\}$. It follows that $U_{2k}[i] - U_0[i] \geq k$. Since $U_0[3] - U_{2k}[3] = 2k$, we conclude that U_0 and U_{2k} are k -crossing. This contradiction completes the proof of the claim. \square

Let j be an integer with $1 \leq j \leq r$. Since $A[3] = s + (j-1)k$ for all $A \in \mathcal{B}_j$, we know from Proposition 6 that (a) each of the first two coordinates determines a linear order on \mathcal{B}_j , and (b) these two linear orders are dual. In particular, if $2 \leq j \leq r-1$, then there is a unique $i \in \{1, 2\}$ for which $X_j[i] > Y_j[i]$.

Now let $i \in \{1, 2\}$. An interval $B = [p, t]$ of consecutive integers from $[1, r-1]$ is called a *block of type i* when the following conditions are satisfied:

- (i) $p = 1$ or $X_p[i] < Y_p[i]$;
- (ii) $X_j[i] > Y_j[i]$ for all $j \in (p, t]$;
- (iii) $t = r-1$ or $X_{t+1}[i] < Y_{t+1}[i]$.

The blocks of type i form a partition of the integer interval $[1, r-1]$. In particular, every $j \in [1, r-1]$ belongs to two blocks, one of each type. Moreover, for every $j \in [1, r-2]$ there is a unique i such that j and $j+1$ belong together to a block of type i . This implies that there are exactly r blocks altogether. When $r = 2$, the singleton set $\{1\}$ is a block of both types, as the three conditions listed above are satisfied vacuously, and it is counted twice.

Choose j with $1 \leq j \leq r-1$. Let $\mathcal{P}_j = (U_0, \dots, U_k)$. For $i \in \{1, 2\}$, let B_i be the block of type i containing j , that is, $B_i = [p_i, t_i]$ with $p_i \leq j \leq t_i$. When a short edge (U_m, U_{m+1}) with $0 \leq m \leq k-1$ is expanded in coordinate i , we say that (U_m, U_{m+1}) is *expanded in B_i* . Each of the short edges (U_m, U_{m+1}) , for $0 \leq m \leq k-1$, is expanded in at least one of B_1 and B_2 .

Now, choose j with $1 \leq j \leq r$. Let U and V be distinct vectors in \mathcal{B}_j that occur consecutively in the two linear orders induced by coordinates 1 and 2. We say that the pair (U, V) *contributes a space* to a block $B = [p, t]$ of type i when one of the following three conditions is satisfied:

- (i) $j = p$ and $U[i] > V[i] \geq Y_j[i]$;
- (ii) $p+1 \leq j \leq t$ and $X_j[i] \geq U[i] > V[i] \geq Y_j[i]$;
- (iii) $j = t+1$ and $X_j[i] \geq U[i] > V[i]$.

Claim 10. *Let j be an integer with $1 \leq j \leq r$. If $U, V \in \mathcal{B}_j$ are consecutive in the linear orders on \mathcal{B}_j determined by coordinates 1 and 2, then exactly one of (U, V) and (V, U) contributes a space to a block and that block is unique.*

Proof. Assume without loss of generality that $U[1] > V[1]$ and $V[2] > U[2]$.

Suppose first that $j = 1$. If $U[1] > V[1] \geq Y_1[1]$, then (U, V) contributes a space to the block of type 1 containing 1. Otherwise, we have $V[2] > U[2] \geq Y_1[2]$ and (V, U) contributes a space to the block of type 2 containing 1.

The proof for the case $j = r$ is similar. If $X_r[1] \geq U[1] > V[1]$, then (U, V) contributes a space to the block of type 1 containing $r - 1$. Otherwise, (V, U) contributes a space to the block of type 2 containing $r - 1$.

Now suppose $2 \leq j \leq r - 1$. There is a unique block B containing both $j - 1$ and j . Assume without loss of generality that B is a block of type 1. If $X_j[1] \geq U[1] > V[1] \geq Y_j[1]$, then (U, V) contributes a space to B . If $U[1] > V[1] \geq X_j[1]$, then (V, U) contributes a space to the block of type 2 containing $j - 1$. Finally, if $Y_j[1] \geq U[1] > V[1]$, then (V, U) contributes a space to the block of type 2 that contains j . \square

Claim 11. *For every block B , the total number of pairs that contribute a space to B and short edges that expand in B is at most $k - 1$.*

Proof. Let $B = [p, t]$ be a block of type i . For $p \leq j \leq t + 1$, let $V_j^0, \dots, V_j^{n_j}$ be the vectors V from \mathcal{B}_j such that

- (i) $V[j] \geq X_j[j]$ when $j \geq p + 1$,
- (ii) $V[j] \leq Y_j[j]$ when $j \leq t$.

Assume further that $V_j^0, \dots, V_j^{n_j}$ are ordered so that $V_j^0[j] > \dots > V_j^{n_j}[j]$. Thus $V_j^{n_j} = X_j$ for $p + 1 \leq j \leq t + 1$, and $V_j^{n_j} = Y_j$ for $p \leq j \leq t$. Clearly, the pairs (V_j^m, V_j^{m+1}) with $p \leq j \leq t + 1$ and $0 \leq m \leq n_j - 1$ are exactly the pairs that contribute a space to B .

For $p \leq j \leq t$, let $\mathcal{P}_j = (U_j^0, \dots, U_j^k)$. Thus $U_j^0 = X_{j+1}$, $U_j^k = Y_j$, and $U_j^k[j] \geq \dots \geq U_j^0[j]$. Clearly, the short edges (U_j^m, U_j^{m+1}) with $p \leq j \leq t$, $0 \leq m \leq k - 1$, and $U_j^{m+1}[i] > U_j^m[i]$ are exactly the short edges that expand in B .

To conclude, since we have

$$\begin{aligned} V_p^0[i] &> \dots > V_p^{n_p}[i] = Y_p[i] = U_p^k[i] \geq \dots \geq U_p^0[i] = X_{p+1}[i] = \\ V_{p+1}^0[i] &> \dots > V_{p+1}^{n_{p+1}}[i] = Y_{p+1}[i] = U_{p+1}^k[i] \geq \dots \geq U_t^0[i] = X_{t+1}[i] = \\ V_{t+1}^0[i] &> \dots > V_{t+1}^{n_{t+1}}[i], \end{aligned}$$

the total number of pairs that contribute a space to B and short edges that expand in B is at most $V_p^0[i] - V_{t+1}^{n_{t+1}}[i]$. Since $V_{t+1}^{n_{t+1}}[3] - V_p^0[3] = (t - p + 1)k$, we have $V_p^0[i] - V_{t+1}^{n_{t+1}}[i] \leq k - 1$, as otherwise V_p^0 and $V_{t+1}^{n_{t+1}}$ would be k -crossing. \square

We are now ready to assemble this series of claims and complete the proof that $|\mathcal{A}_s| \leq k$. For $1 \leq j \leq r$, let $b_j = |\mathcal{B}_j|$. Thus $|\mathcal{A}_s| = b_1 + \dots + b_r$. By Claim 10, there are $b_j - 1$ ordered pairs of elements from \mathcal{B}_j that occur consecutively in the linear orders on \mathcal{B}_j induced by coordinates 1 and 2 and each contributes a space to one of the r blocks. Also, each of the $(r - 1)k$ short edges on the paths $\mathcal{P}_1, \dots, \mathcal{P}_{r-1}$ is expanded in at least one block. Thus by Claim 11 we have

$$\sum_{j=1}^r (b_j - 1) + (r - 1)k \leq r(k - 1).$$

On the other hand, we have

$$\sum_{j=1}^r (b_j - 1) = |\mathcal{A}_s| - r.$$

Composing the two we obtain $|\mathcal{A}_s| \leq k$, which completes the proof of Theorem 3.

5. GENERALIZATION

For $w \geq 1$ and $1 \leq k_1 \leq \dots \leq k_w$, we say that vectors A and B from \mathbb{Z}^w are (k_1, \dots, k_w) -crossing when there are two coordinates i and j for which $A[i] - B[i] \geq k_i$ and $B[j] - A[j] \geq k_j$. Let $f(k_1, \dots, k_w; w)$ denote the maximum size of a subset of \mathbb{Z}^w with every two vectors being 1-crossing but with no two (k_1, \dots, k_w) -crossing vectors. Thus $f(k, w) = f(k, \dots, k; w)$.

Proposition 12. *For $w \geq 1$ and $k_1, \dots, k_w \geq 1$ we have*

$$k_2 \cdots k_w \leq f(k_1, \dots, k_w; w) \leq k_1 \cdots k_w.$$

The proof of Proposition 12 follows along the same lines as the proof of the inequalities $k^{w-1} \leq f(k, w) \leq k^w$ at the beginning of Section 3. We propose a conjecture which seems to be more general but turns out to be equivalent to Conjecture 1.

Conjecture 13. *For $w \geq 1$ and $1 \leq k_1 \leq \dots \leq k_w$ we have*

$$f(k_1, \dots, k_w; w) = k_2 \cdots k_w.$$

Proposition 14. *Conjectures 1 and 13 are equivalent.*

Proof. Clearly, Conjecture 13 yields Conjecture 1. To prove the converse implication, we suppose $f(k_1, \dots, k_1; w) = k_1^{w-1}$ and prove $f(k_1, \dots, k_w; w) = k_2 \cdots k_w$. Let \mathcal{A} be an antichain in \mathbb{Z}^w with no two (k_1, \dots, k_w) -crossing vectors. For any selection of k_1 -element subsets $I_2 \subseteq \{0, \dots, k_2 - 1\}$, ..., $I_w \subseteq \{0, \dots, k_w - 1\}$, consider the family $\mathcal{A}(I_2, \dots, I_w) = \{A \in \mathcal{A} : A[i] \bmod k_i \in I_i \text{ for } 2 \leq i \leq w\}$. Now modify each $A \in \mathcal{A}(I_2, \dots, I_w)$ to get a vector A' so that if $A[j] = a_j k_j + r_j$, where $0 \leq r_j < k_j$, and ℓ_j is the position of r_j in a natural ordering of I_j , then $A'[i] = a_j k_1 + \ell_j$. Clearly, the resulting family \mathcal{A}' of all the vectors A' is an antichain and contains no two k_1 -crossing vectors. Thus $|\mathcal{A}(I_2, \dots, I_w)| = |\mathcal{A}'| \leq k_1^{w-1}$. Summing up over all selections of subsets I_2, \dots, I_w we obtain

$$\binom{k_2-1}{k_1-1} \cdots \binom{k_w-1}{k_1-1} |\mathcal{A}| \leq \binom{k_2}{k_1} \cdots \binom{k_w}{k_1} k_1^{w-1},$$

$$|\mathcal{A}| \leq k_2 \cdots k_w. \quad \square$$

Proposition 14 tells us that in some sense the most difficult case is when all k_i are equal. Surprisingly, for some values of k_i we know the exact answer. For instance,

$$f(k, k, 2k, \dots, 2^{w-1}k; w) = k \cdot 2k \cdots 2^{w-1}k.$$

Namely, we show that

$$f(k, k, 2k, \dots, 2^{w-1}k; w) \leq k f(2k, 2k, \dots, 2^{w-1}k; w-1),$$

which together with $f(k, k; 2) = k$ and Proposition 12 gives the previous equality. We define $A <_{12} B$ if $A[1] < B[1]$ and $A[2] > B[2]$. Every maximum chain in this order has size at most k , as otherwise it would yield a $(k, k, 2k, \dots, 2^{w-1}k)$ -crossing. Let \mathcal{A}' be a family of vectors of a fixed height in the order $<_{12}$. Now let $\phi(A) = (A[1] + A[2], A[3], \dots, A[w])$ for $A \in \mathcal{A}'$. The mapping ϕ is an injection, and $\phi(\mathcal{A}')$ is an antichain of pairwise non- $(2k, 2k, \dots, 2^{w-1}k)$ -crossing vectors from \mathbb{Z}^{w-1} . This gives the required inequality.

6. EXTREMAL EXAMPLES

Some classic extremal problems have elegant solutions due to the fact that all maximal structures are also maximum. For example, the maximum number of edges in a planar graph is $3n - 6$ when $n \geq 3$, because if G is any planar graph containing a face that is not a triangle, then an edge can be added to G while preserving planarity.

Other extremal problems may have many different maximal structures, but essentially only one which is maximum. An example of this is Turán's theorem which asserts that the maximum number of edges in a graph on n vertices which does not contain a complete subgraph on $k + 1$ vertices is the number of edges in the complete k -partite graph on n vertices, where the part sizes are as balanced as possible. Another example is Sperner's theorem which asserts that the only maximum antichains in the lattice of all subsets of $\{1, \dots, n\}$ are the ranks at levels $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.

It is our feeling that the extremal problem discussed in this paper is challenging because there are many different examples that we suspect to be extremal. We already presented one example at the beginning of Section 3, and in this section we develop some others.

6.1. Inductive construction. Suppose that we have constructed an antichain \mathcal{A} of pairwise non- k -crossing vectors in \mathbb{Z}^w , and suppose it is contained in $[0, c)^w$. We are going to construct an antichain \mathcal{A}' of size $k|\mathcal{A}|$ on $w + 1$ coordinates. Put k disjoint copies of \mathcal{A} one above another on coordinates $1, \dots, w$, that is, the i th copy inside $[(i - 1)c, ic)^w$, and set the coordinate $w + 1$ to be $-i$ for all vectors in the i th copy. This way we obtain a k times larger antichain of pairwise non- k -crossing vectors in \mathbb{Z}^{w+1} . If $|\mathcal{A}| = k^{w-1}$, then $|\mathcal{A}'| = k^w$.

6.2. Lexicographic construction. When $A \in \mathbb{Z}^w$, the *rank* of A is the sum $A[1] + \dots + A[w]$. Let $k, w \geq 2$. We construct an antichain \mathcal{A} in \mathbb{Z}^w as follows. First, consider the family \mathcal{F} of all vectors in \mathbb{Z}^w with $0 \leq A[i] \leq k - 1$ for $1 \leq i \leq w$ and $\sum_{i=1}^w A[i] \equiv w(k - 1) \pmod{k}$. Clearly, there are k^{w-1} vectors in \mathcal{F} . For each $A \in \mathcal{F}$, there is a unique non-negative integer $m(A)$ such that

$$m(A) \cdot k + A[1] + \dots + A[w] = w(k - 1).$$

Let n be the maximum value of $m(A)$ taken over all vectors $A \in \mathcal{F}$. Then let $\tau = (i_1, \dots, i_n)$ be any sequence of integers from $\{1, \dots, w\}$. We modify \mathcal{F} into an antichain \mathcal{A} by the following rule. If $A \in \mathcal{F}$, then we modify A by increasing coordinate i by pk , where p is the number of times i occurs at the first $m(A)$ positions of τ . Clearly, these modifications result in a family \mathcal{A} consisting of k^{w-1} vectors. Furthermore, since each vector $A \in \mathcal{A}$ has rank $w(k - 1)$, we know that \mathcal{A} is an antichain. Also, \mathcal{A} has no k -crossing pair.

The example presented at the beginning of Section 3 is the special case of this construction where τ is the constant sequence (w, \dots, w) .

6.3. Cyclic construction. Here, we fix $w = 3$ and consider coordinates $\{1, 2, 3\}$ in clockwise order. Thus if $i = 3$ then $i + 1 = 1$, and if $i = 1$ then $i - 1 = 3$. Let $k \geq 2$. If $k \equiv 0 \pmod{3}$ or $k \equiv 2 \pmod{3}$, then the family \mathcal{A} defined by

$$\begin{aligned} \mathcal{A} = \{A \in \mathbb{Z}^3 : \text{rank}(A) = 2k - 1, A[i + 1] \leq k + A[i], \\ \text{and } A[i - 1] \leq k - 1 + A[i] \text{ for } i \in \{1, 2, 3\}\} \end{aligned}$$

is an antichain, contains no k -crossing pair, and has size k^2 . In these two cases, there is clearly a cyclic symmetry between all three coordinates.

If $k \equiv 0, 2 \pmod{3}$, then we get k^2 vectors. However, when $k \geq 4$ and $k \equiv 1 \pmod{3}$, this construction yields only $k^2 - 1$ vectors. In the latter case, we can add the single vector $(0, k+1, k-2)$ to form a maximum family with no k -crossing pair, but we lose the cyclic symmetry.

On the other hand, if we consider $k \equiv 1 \pmod{3}$ and take vectors of rank $2k-2$, then we get a cyclic family of size k^2 .

6.4. Remarks on rank. All the examples we have constructed so far are *ranked* antichains, that is, they consist of vectors in \mathbb{Z}^w all of which have the same rank. Based on this observation, it would be tempting to try to reduce the entire problem to ranked antichains. Indeed, we have the following proposition.

Proposition 15. *For all $k, w \geq 1$, the maximum size of a ranked antichain in \mathbb{Z}^w containing no k -crossing pair is k^{w-1} .*

Proof. We only need to prove that if \mathcal{A} is a ranked antichain in \mathbb{Z}^w , then $|\mathcal{A}| \leq k^{w-1}$, and we may assume as before that $k, w \geq 2$. For each vector A in \mathcal{A} , let $\sigma(A)$ denote the vector in $\{0, \dots, k-1\}^{w-1}$ such that $A[i] \equiv \sigma(A)[i] \pmod{k}$ for $1 \leq i \leq w-1$. Clearly, σ is an injection and its range has at most k^{w-1} elements. \square

However, we know the following example.

Example 16. *For $k = 2$ and $w = 4$, the following 8 vectors form a non-ranked antichain in \mathbb{Z}^4 with no 2-crossing pair:*

$$\begin{aligned} (0, 2, 1, 1), & \quad (2, 1, 0, 1), & (1, 0, 2, 1), & \quad (1, 1, 1, 1), \\ (1, 3, 2, 0), & \quad (3, 2, 1, 0), & (2, 1, 3, 0), & \quad (2, 2, 2, 0). \end{aligned}$$

Moreover, this antichain is compressed on each of the four coordinates.

The first four of the vectors above have rank 4, while the last four have rank 6. More generally, any family obtained by the cyclic construction (Subsection 6.3) can be extended to $w = 4$ in an analogous manner.

6.5. Remarks on the size of the largest coordinate.

Proposition 17. *Let k and w be positive integers. Suppose that \mathcal{A} is an antichain in \mathbb{Z}^w of pairwise non- k -crossing vectors, and suppose that there is a coordinate j on which all vectors are different. Then $|\mathcal{A}| \leq k^{w-1}$.*

Proof. Suppose all vectors differ on the first coordinate. For $2 \leq i \leq w$, we define an order $<_i$ on \mathcal{A} as follows. We put $A <_i B$ if $A[1] < B[1]$ and $A[i] > B[i]$. The maximum size of a chain in this order is at most k , as otherwise we would have a k -crossing. Let $\phi(A) \in \{1, \dots, k\}^{w-1}$ be the vector of heights of A in orders $<_2, \dots, <_w$. Clearly, if $A, B \in \mathcal{A}$ are such that $A[1] < B[1]$, then for some coordinate i we have $A[i] > B[i]$, and thus the heights of A and B in $<_i$ are different. This shows that the mapping $\phi : A \rightarrow \{1, \dots, k\}^{w-1}$ is injective. \square

It follows that in any antichain of pairwise non- k -crossing vectors in \mathbb{Z}^w and any coordinate the number of different values attained on it is at most k^{w-1} . Otherwise, a choice of representatives of the attained values would contradict Proposition 17.

6.6. Remarks on compression. Careful analysis of the proof of the case $w = 3$ shows that we do not really need a fully compressed coordinate. We only use the following two properties:

- (P₁) For $1 \leq j \leq r$ the set $\mathcal{B}_j = \{A \in \mathcal{A} : A[3] = i + k(j-1)\}$ is an antichain.
- (P₂) For $1 \leq j \leq r-1$ there is a short chain from a vector X_{j+1} in \mathcal{B}_{j+1} to a vector Y_j in \mathcal{B}_j .

However, we know that when $w = 4$ this weaker notion of compression is not enough. To see this, consider the union of the following families of vectors in \mathbb{Z}^4 :

- (i) The vectors for which $0 \leq A[1], A[2] \leq k - 1$, $A[3] \geq 2$, $A[1] + A[2] + A[3] = 2k - 2$, and $A[4] = k$.
- (ii) All vectors of the form $(i, k - 1 - i, k + 1, 0)$ where $0 \leq i \leq k - 1$.
- (iii) The vector $(k - 1, k - 1, k, 0)$.
- (iv) All vectors having rank $3k - 2$ with $1 \leq A[i] \leq k - 1$ for $1 \leq i \leq 4$.

It is easy to see that this family satisfies properties (P_1) and (P_2) but has more than k^2 vectors for which $A[4] \equiv 0 \pmod{k}$.

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